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Inverse bremsstrahlung absorption in large radiation fields during binary collisions—classical theory II. Integrated rate coefficients for Coulomb collisions

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Abstract. Integrated rate coefficients for the absorption of electromagnetic radiation in plasma when the plasma frequency is less than the radiation frequency are calculated from classical expressions derived earlier. The resulting formula is applicable over the complete range of field strengths. Numerical calculations are used to give useful interpolation formulae for the complete range of field strengths. The behaviour at high field strengths is discussed.

1. Introduction

The continued interest in high-field inverse bremsstrahlung has led to a number of papers in which the photon absorption cross section has been calculated as a function of the electron velocity. However, few expressions for the total integrated rate in a plasma have been given. Osborn (1972) has derived from the Born approximation result an expression involving a double sum containing modified Bessel functions of the second kind, which do not rapidly converge when a large number of photons are absorbed in each collision. Silin (1965) using a classical approach has given some general results, but in their simpler forms these appear to apply only to restricted physical conditions.

In this paper the classical approximation discussed earlier (Pert 1972a) is used to derive an integrated expression for the total radiation and absorption rate by inverse bremsstrahlung of a plane-polarized beam of arbitrary intensity in a fully ionized plasma.

Numerical calculations of this general, but cumbersome, formula, are used to develop simple polynomial approximations valid over a limited range. A simple representation of questionable validity is used to investigate the appropriate form of the absorption coefficient at high fields. However, the good fit shown by the results obtained numerically to this form confirms the usefulness of this approximate analysis.

2. The total absorption rate

We consider a plane-polarized beam of electromagnetic radiation of frequency, ω, propagating in a plasma whose plasma frequency $\omega_p \ll \omega$. We have shown earlier that the absorption rate is determined by the electron 'quiver' velocity

$$\boldsymbol{u} = e\boldsymbol{E}/m\omega \tag{1}$$

where $\boldsymbol{E}$ is the instantaneous electric field, e and m the electronic charge and mass respectively. The average energy gain per electron of thermal velocity $\boldsymbol{v}_{\mathrm{T}}$ in time δt is given by

$$mn_{\mathrm{i}}\,\overline{(\boldsymbol{u}\cdot\boldsymbol{v}v\sigma_{\mathrm{d}})}\,\delta t \tag{2}$$

where $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{v}_{\mathrm{T}}$ is the total velocity, and σ_{d} the momentum transfer cross section: the average is taken over time and over the angular distribution function.

The role of high fields perturbing the electron distribution is discussed in Pert (1972a, b). It is concluded that in a fully-stripped plasma where Coulombic collisions dominate, electron–electron collisions will be much more frequent and that in consequence a Maxwellian distribution will be maintained. In this paper we therefore assume that the electron distribution may be satisfactorily represented by a Maxwellian of temperature T_{e}:

$$f(\boldsymbol{v}_{\mathrm{T}}) = (m/2\pi kT_{\mathrm{e}})^{3/2}\exp(-mv_{\mathrm{T}}^2/2kT_{\mathrm{e}}). \tag{3}$$

We may write the instantaneous energy absorption rate averaged over the electron distribution

$$R = mn_{\mathrm{i}}\int f(\boldsymbol{v}_{\mathrm{T}})\sigma_{\mathrm{d}}(v)v\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}^3\boldsymbol{v}_{\mathrm{T}}. \tag{4}$$

Thus changing the integration to one over $\boldsymbol{v}$ rather than $\boldsymbol{v}_{\mathrm{T}}$ we have:

$$R = 2\pi mn_{\mathrm{i}}\gamma(m/2\pi kT_{\mathrm{e}})^{3/2}\int\int \boldsymbol{u}\cdot\boldsymbol{v}v^{-1}\ln(\alpha v^2)\,\mathrm{d}v\,\mathrm{d}(\cos\theta)\exp[-m(\boldsymbol{v}-\boldsymbol{u})^2/2kT_{\mathrm{e}}] \tag{5}$$

where σ_{d} for Coulomb collisions is given by:

$$\sigma_{\mathrm{d}} = \gamma\ln(\alpha v^2)/v^4 \tag{6}$$

where

$$\gamma \to \frac{4\pi e^4 Z^2}{m^2 v^4} \qquad \text{and} \qquad \ln(\alpha v^2) \to \ln\left(\frac{v/w}{Ze^2/mv^2}\right).$$

Performing the integral over the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$ and averaging over time we obtain

$$\begin{aligned}\bar{R} = {} & 2n_{\mathrm{i}}kT_{\mathrm{e}}(m/2\pi kT_{\mathrm{e}})^{3/2}\int \mathrm{d}v\,\sigma_{\mathrm{d}}(v)\exp[m(u_0^2+v^2)/2kT_{\mathrm{e}}]\sum_{l=0}^{\infty}2\pi^{1/2}\Gamma(l+\tfrac{1}{2})\\ & \times(mu_0^2/2kT_{\mathrm{e}})^l/l!\Big((2kT_{\mathrm{e}}/mu_0v)^l\mathrm{I}_l(mu_0v/kT_{\mathrm{e}})\\ & -(kT_{\mathrm{e}}/mu_0v)\int_0^{mu_0v/kT_{\mathrm{e}}}(2/\beta)^l\mathrm{I}_l(\beta)\,\mathrm{d}\beta\Big)\end{aligned} \tag{7}$$

where u_0 is the amplitude of the 'quiver' velocity, and I_l a modified Bessel function of the first kind. This expression may be simplified to give:

$$\bar{R}=2n_i m(m/2\pi kT_e)^{1/2}\sum_{n=1}^{\infty}[2n/(2n+1)][1/(n!)^2]M(\tfrac{1}{2},n+1,mu_0^2/2kT_e)$$
$$\times\int dv(\gamma/v)\ln(\alpha v^2)(mu_0 v/2kT_e)^{2n}\exp(-mv^2/2kT_e) \tag{8}$$

where M is a confluent hypergeometric function (Slater 1960). The integration over v may be performed in a standard manner to give

$$\bar{R}=2n_i m\gamma(m/2\pi kT_e)^{1/2}\sum_{n=1}^{\infty}[1/(2n+1)(1/n!)(mu_0^2/2kT_e)^n\exp(-mu_0^2/2kT_e)$$
$$\times M(\tfrac{1}{2},n+1,mu_0^2/2kT_e)[\psi(n)+\ln(2\alpha kT_e/m)] \tag{9}$$

where $\psi(n)$ is the digamma function.

The absorption coefficient is given by:

$$\kappa=\bar{R}n_e/(cE^2/8\pi)^{-1}$$
$$=\frac{8\pi n_e n_i Z^2 e^6}{cv^2(2\pi mkT_e)^{3/2}}\sum_{n=1}^{\infty}\frac{1}{n!(2n+1)}\left(\frac{mu_0^2}{2kT_e}\right)^{n-1}\exp(-mu_0^2/2kT_e)$$
$$\times M(\tfrac{1}{2},n+1,mu_0^2/2kT_e)[\ln(8k^3T_e^3/Z^2\omega^2e^4m)+3\psi(n)] \tag{10}$$

where n_e is the electron density, c the velocity of light and $\nu=\omega/2\pi$ is the radiation frequency.

The coefficient of the digamma function—in this case 3—depends on the nature of the lower cut-off used. We have here assumed that the cut-off is given by the Landau parameter determined by v. If a constant impact parameter cut-off is used as by Dawson and Oberman (1962) the factor is 1 and the first-order term $n=1$ is identical to their expression. In general if the ratio of the impact parameters $(b_{\max}/b_{\min})$ varies as v^η the coefficient of the digamma function is η.

A small-order expansion of (10) may be easily made using Kummer's transformation to give:

$$\kappa=\frac{8\pi n_e n_i Z^2 e^6}{3cv^2(2\pi mkT_e)^{3/2}}\left\{\left[\ln\left(\frac{8k^3T_e^3}{Z^2\omega^2e^4m}\right)-\eta\gamma\right](1-\tfrac{9}{20}x+\tfrac{15}{128}x^2-\tfrac{35}{1152}x^3+\tfrac{21}{11264}x^4+\ldots)\right.$$
$$\left.+\eta(\tfrac{3}{10}x-\tfrac{1}{7}x^2+\tfrac{71}{1728}x^3-\tfrac{31}{3520}x^4+\ldots)\right\}. \tag{11}$$

3. Standard formulae

In calculations of laser–plasma interaction it is convenient to have simple formulae for the absorption coefficient which can be used with a minimum of computation. Due to the presence of the sums, equation (10) is clearly not of this form. We may, however, obtain a 'universal' formula by writing (10) in the form:

$$\kappa=\frac{8\pi n_e n_i Z^2 e^6}{3cv^2(2\pi mkT_e)^{3/2}}\left[\mathcal{S}_1\left(\frac{mu_0^2}{2kT_e}\right)\ln\left(\frac{8k^3T_e^3}{Z^2\omega^2e^4m}\right)+\eta\mathcal{S}_2\left(\frac{mu_0^2}{2kT_e}\right)\right] \tag{12}$$

where

$$\mathscr{S}_1(x) = 3 \sum_{n=1}^{\infty} [1/(2n+1)]x^{n-1}[M(\tfrac{1}{2}, n+1, x)/n!]\,\mathrm{e}^{-x}$$

and

$$\mathscr{S}_2(x) = 3 \sum_{n=1}^{\infty} [1/(2n+1)]x^{n-1}[M(\tfrac{1}{2}, n+1, x)/n!]\psi(n)\,\mathrm{e}^{-x} \tag{13}$$

provided we can find a convenient representational form for $\mathscr{S}_1$ and $\mathscr{S}_2$†. This was done by evaluating the sums $\mathscr{S}_1$ and $\mathscr{S}_2$ directly. For large values of x this gave considerable trouble due to the change in the asymptotic representation in the region $x \simeq n$. In this region the sum had to be numerically evaluated by direct summation of confluent hypergeometric series, which limited the range of x to values less than 5×10^3, due to problems associated with finite numerical range of the computer and round off errors in the sum. An alternative representation using direct integration of (5) was tried but found unsatisfactory due to the finite range of the exponential available on the computer.

The evaluated sums are shown plotted in figure 1. It may be clearly seen that a significant decrease in the absorption coefficient occurs, when $mu_0^2/2kT_e > 1$, i.e. when the 'quiver' velocity exceeds the mean thermal velocity as discussed previously.

For computational purposes we have obtained a least-squares fit for $\mathscr{S}_1$ and $\mathscr{S}_2$ as functions of x, valid for $0{\cdot}1 < x < 10{\cdot}0$.

For large x we have used a representation for $\mathscr{S}_1$ and $\mathscr{S}_2$ suggested by the analysis in the next section

$$S_1 = (\surd\pi)x^{3/2}\mathscr{S}_1/3 = \tfrac{1}{2}\ln x + a_0 + a_1/x + a_2/x^2 + \ldots$$

and

$$S_2 = (\surd\pi)x^{3/2}\mathscr{S}_2/3 = (S_1 + \alpha)^2 + b_0 + b_1/x + b_2/x^2 + \ldots \tag{14}$$

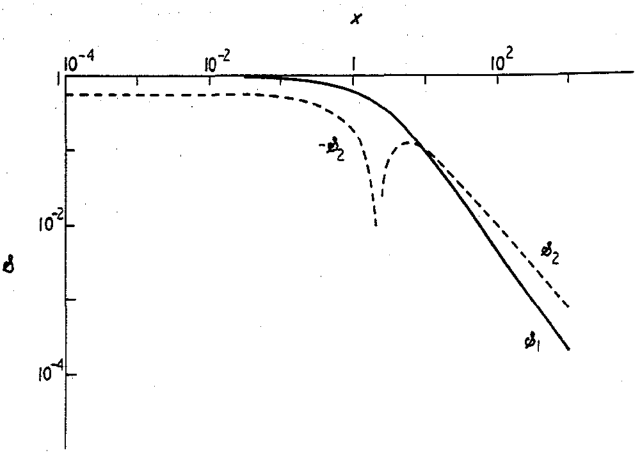

Figure 1. Plot of the sums $\mathscr{S}_1$ and $\mathscr{S}_2$, equation (20) as functions of x. Note that the plot is logarithmic. Thus the zero of $\mathscr{S}_2$ at $x \simeq 2{\cdot}1483$ appears as a discontinuity: for small values of x, $\mathscr{S}_2$ is negative and for large values positive.

† The factor 3 appearing in (12) and (13) ensures the limits:

$$\lim_{x\to 0} \mathscr{S}_1 = 1; \qquad \lim_{x\to 0} \mathscr{S}_2 = -\gamma.$$

the coefficients $a_0, a_1 \ldots, b_0, b_1 \ldots$ being found by a least-squares fit. It was found that the constant α gave a significant improvement for the value of S_2. We may compare these values with those given by the subsequent analysis. The value of the constant, α, found by a least-squares fit was 0·018 275 3, which is in good agreement with the calculated value of 0·018 244 99, $(1-\ln 2-\frac{1}{2}\gamma)$, given by equation (25). This agreement was to be expected as the principal contribution to this constant comes from contributions to the sum (10) with small n. The values of a_0 and b_0 are not, however, compatible with equations (25) and (26) due, it is believed, to the higher-order terms in the asymptotic expansion, which cause non-exponential like behaviour in the region $n \sim x$. The results obtained were consistent with estimates made of the deviations incurred by this behaviour.

Thus our standard formulae for the general absorption coefficient may be summarized:

$$\kappa=\frac{8\pi n_e n_i Z}{3cv^2(2\pi mkT_e)^{3/2}}[(\ln\Delta-\eta\gamma)(1-\tfrac{9}{20}x+\tfrac{15}{128}x^2-\tfrac{35}{1152}x^3+\tfrac{21}{11264}x^4)$$
$$+\eta(\tfrac{3}{10}x-\tfrac{1}{7}x^2+\tfrac{71}{1728}x^3-\tfrac{31}{3520}x^4)] \qquad (15)$$

in the range $0<x<0{\cdot}1$.

$$\kappa=\frac{8\pi n_e n_i Z}{3cv^2(2\pi mkT_e)^{3/2}}[\ln\Delta(9{\cdot}7841\times10^{-1}-3{\cdot}5838\times10^{-1}x+6{\cdot}0991\times10^{-2}x^2$$
$$-4{\cdot}7356\times10^{-3}x^3+1{\cdot}3266\times10^{-4}x^4)$$
$$-\eta(5{\cdot}3799\times10^{-1}-3{\cdot}9001\times10^{-1}x+8{\cdot}0196\times10^{-2}x^2$$
$$-6{\cdot}7431\times10^{-3}x^3-1{\cdot}9651\times10^{-4}x^4)] \qquad (16)$$

in the range $0{\cdot}1<x<10{\cdot}0$‡.

$$\kappa=\frac{64\pi^2 n_e n_i e^3 Z^2 v}{cE^3}(S_1\ \ln\Delta+S_2\eta) \qquad (17)$$

in the range $10<x<\infty$ where

$$S_1=\tfrac{1}{2}\ln x+0{\cdot}637\,78+17{\cdot}3545/x-894{\cdot}685/x^2+11\,792{\cdot}6/x^3-46\,022{\cdot}2/x^4$$

and

$$S_2=(S_1+0{\cdot}018\,2753)^2-1{\cdot}329\,35-39{\cdot}2639/x+1804{\cdot}26/x^2-17\,135{\cdot}8/x^3$$
$$+30\,532{\cdot}2/x^4. \qquad (18)$$

In these formulae Δ and x have the following values

$$\Delta=(8k^3T_e^3/Z^2\omega^2e^4m) \qquad \text{and} \qquad x=mu_0^2/2kT_e. \qquad (19)$$

‡ An alternative more accurate, but higher-order, expression for (16) is:

$$\kappa=\frac{8\pi n_e n_i Z}{3cv^2(2\pi mkT_e)^{3/2}}[\ln\Delta(9{\cdot}9310\times10^{-1}-4.1184\times10^{-1}x+9{\cdot}2852\times10^{-2}x^2-1{\cdot}150\,13\times10^{-2}x^3$$
$$+7{\cdot}1657\times10^{-4}x^4-1{\cdot}726\,19\times10^{-5}x^5)$$
$$-\eta(5{\cdot}634\,04\times10^{-1}-4{\cdot}824\,79\times10^{-1}x+1{\cdot}353\,13\times10^{-1}x^2-1{\cdot}844\,76\times10^{-2}x^3$$
$$+1{\cdot}206\,64+10^{-3}x^4+2{\cdot}986\,21\times10^{-5}x^5)].$$

These formulae should be accurate to better than 1 part in 10^3. However, it should be remembered that their validity is limited by physical restrictions on the model used, namely binary collisional behaviour and the use of simple cut-offs for the cross section, which probably introduce errors larger than the computational ones. It should also be noted that only equation (15) is a true expansion of (10), the others being derived by calculating the best polynomial fit to the numerical values obtained by evaluating the sum directly. In particular, equation (18) should not be regarded as an asymptotic expansion, although it is probable the true one will have the same form with different numerical coefficients.

4. High-field limit

Unfortunately, it is not possible to derive the true asymptotic limit of equation (10) at high fields, since the complete asymptotic expansion for $M(a, b, x)$ for large x is not known. However, it is known (Slater 1960) that for large x

$$M(a, b, x) \simeq \frac{\Gamma(b)}{\Gamma(a)} x^{-(b-a)} e^{x} \qquad b \ll x \tag{20}$$

and

$$M(a, b, x) \simeq (1 - x/b)^{-a}\{1 - [a(a+1)/2b][x/(b-x)]^2\} \qquad b \gg x \gg 1. \tag{21}$$

Thus the term appearing in the sum (10) has the limiting forms for large x:

$$\lim_{n\to 0} \frac{\Gamma(\frac{1}{2}) x^{n+\frac{1}{2}} e^{-x}}{\Gamma(n+1)} M(\tfrac{1}{2}, n+1, x) \simeq 1$$

$$\lim_{n\to \infty} \frac{\Gamma(\frac{1}{2}) x^{n+\frac{1}{2}} e^{-x}}{\Gamma(n+1)} M(\tfrac{1}{2}, n+1, x) \simeq \frac{\Gamma(\frac{1}{2}) x^{n} e^{-x}}{\Gamma(n+1)} \to 0.$$

This suggests that in order to investigate the form of the asymptotic limit of the sum we replace the confluent hypergeometric function term in (10) by a cut-off term. In doing this it must be clearly born in mind that we do not regard the result thus obtained as an asymptotic limit, but rather as a guide to its form. Confirmation that this guess provides an appropriate form of the asymptotic solution is shown by the good fit of the numerically calculated solution to this form. For this purpose we have used a fairly general exponential cut-off:

$$\frac{\Gamma(\frac{1}{2}) x^{n+\frac{1}{2}} e^{-x}}{\Gamma(n+1)} M(\tfrac{1}{2}, n+1, x) \sim \exp[-(n/\xi x)^{\zeta}] \tag{22}$$

the exponential term being chosen with variable coefficients ξ and ζ, which allow the position and 'sharpness' of the cut-off to be varied.

In order to investigate the behaviour at high field we replace the confluent hypergeometric function in (10) by the crude representation (22). In this case the absorption coefficient is given by equation (17) but with the sums S_1 and S_2 replaced by:

$$S_1 = \sum_{n=1}^{\infty} [1/(2n+1)] \exp[-(n/\xi x)^{\zeta}] \tag{23}$$

and

$$S_2 = \sum_{n=1}^{\infty} [\psi(n)/(2n+1)] \exp[-(n/\xi x)^{\zeta}] \tag{24}$$

respectively. The sums have been evaluated in the appendix with ξx replaced by x, giving

$$S_1 = \tfrac{1}{2}[\ln(\xi x) - \gamma/\zeta] - 0{\cdot}018\,244\,99 \tag{25}$$

and

$$S_2 = \tfrac{1}{4}[\ln(\xi x) - \gamma/\zeta]^2 + \pi^2/24\zeta^2 - 0{\cdot}589\,1150. \tag{26}$$

These results were used to suggest the correct form for the high-field expressions, S_1 and S_2, for $\mathcal{S}_1$ and $\mathcal{S}_2$ in equation (14) in the numerical calculations.

In particular for $\xi = 1$ and $\zeta \to \infty$ or from the numerical results we obtain:

$$\kappa = \frac{16\pi n_e n_i e^3 Z^2 w}{cE^3} \left\{ \ln\left(\frac{mu_0^2}{2kT_e}\right) \ln\left(\frac{8k^3 T_e^3}{Z^2 \omega^2 e^4 m}\right) + \frac{\eta}{2}\left[\ln\left(\frac{mu_0^2}{2kT_e}\right)\right]^2 \right\} \tag{27}$$

which is in agreement with the value obtained earlier (Pert 1972a, b) when the digamma factor is neglected in accordance with the constant cut-off ($\eta = 0$), as assumed previously.

5. Conclusions

We have derived an integrated rate coefficient for the absorption of electromagnetic radiation in binary Coulomb collisions which is valid over the complete range of field strengths. This result is clearly limited to studies involving fully-ionized plasma whose plasma frequency is much greater than the radiation frequency. This result involving an infinite sum containing confluent hypergeometric functions will converge rapidly if the electron thermal energy is greater than its 'quiver' energy, and is therefore a more convenient form to use than that due to Osborn (1972) in the classical limit, where the individual photon energy is small and in each collision a large number of photons are simultaneously absorbed. We may note that this represents a sum over a large number of high-order multiphoton processes.

At high fields the sum does not rapidly converge. We have therefore derived the asymptotic expansion for use in this case.

These results do not show any similarity to the general expression derived by Silin (1965), probably due to the complicated nature of this result. Silin has only given simplified results which are valid when $\omega \ll v_T/\lambda_D \simeq \omega_p$, i.e., outside the range of the present theory.

Appendix. On the evaluation of certain sums

We require the evaluation of:

$$S_1 = \sum_{n=1}^{\infty} (2n+1)^{-1} \exp[-(n/x)^{\zeta}] \tag{A.1}$$

for very large values of x. Since $\zeta \gtrsim 1$ we may clearly write

$$S_1 \simeq \lim_{N\to\infty}\left(\sum_{n=1}^{N}(2n+1)^{-1} - \int_{\frac{1}{2}}^{N+\frac{1}{2}}(2n+1)^{-1}\,\mathrm{d}n\right) + \int_{\frac{1}{2}}^{n_1+\frac{1}{2}}(2n+1)^{-1}\,\mathrm{d}n$$
$$+\tfrac{1}{2}\int_{n_1+\frac{1}{2}}^{\infty}\exp[-(n/x)^{\zeta}](\mathrm{d}n/n) \qquad \text{(A.2)}$$

where n_1 is chosen such that $x \gg n_1 \gg 1$. Since

$$\sum_{n=1}^{N}\frac{1}{2n+1} = \sum_{n=1}^{2N+1}\frac{1}{n} - \frac{1}{2}\sum_{n=1}^{N}\frac{1}{n} - 1 = \tfrac{1}{2}[\gamma + \ln(n)] + \ln 2 - 1 \qquad \text{(A.3)}$$

where γ is Euler's constant, we obtain:

$$S_1 \simeq (\tfrac{1}{2}\gamma + \ln 2 - 1) + \tfrac{1}{2}\ln(n_1 + \tfrac{1}{2}) + (2\zeta)^{-1}\mathrm{E}_1(n_1/x)^{\zeta} \qquad \text{(A.4)}$$

where E_1 is the exponential integral. Since $x \gg n_1 \gg 1$ we obtain:

$$S_1 \simeq \tfrac{1}{2}(\ln x - \gamma/\zeta) + \tfrac{1}{2}\gamma + \ln 2 - 1. \qquad \text{(A.5)}$$

The second sum required is:

$$S_2 = \sum_{n=1}^{\infty}[1/(2n+1)]\psi(n)\exp[-(n/x)^{\zeta}]. \qquad \text{(A.6)}$$

In this we use the asymptotic form of $\psi(n)$, namely $\ln(n)$ to write:

$$S_2 \simeq B + \int_{1/2}^{\infty}\frac{\ln(n)\exp[-(n/x)^{\zeta}]}{2n+1}\mathrm{d}n \qquad \text{(A.7)}$$

where

$$B = \lim_{N\to\infty}\left(\sum_{n=1}^{N}\psi(n)/(2n+1) - \int_{1/2}^{N+\frac{1}{2}}[\ln(n)/(2n+1)]\,\mathrm{d}n\right)$$
$$= \lim_{N\to\infty}\left(\sum_{n=1}^{N}\psi(n)/(2n+1) - \tfrac{1}{4}[\ln(N)]^2 - \tfrac{1}{4}[(\ln 2)^2 - \pi^2/6]\right). \qquad \text{(A.8)}$$

The integral in equation (A.7) is evaluated as follows:

$$\int_1^{\infty}\frac{\ln(n)\exp[-(n/x)^{\zeta}]}{n+1}\mathrm{d}n$$
$$\simeq \frac{1}{\zeta^2}\int_1^{\infty}\frac{\ln(n)\exp[-(n/x^{\zeta})]}{n+1}\mathrm{d}n - \frac{1}{\zeta^2}\int_1^{\infty}\ln(n)\left(\frac{1}{n+1} - \frac{1}{n^{(\zeta-1)/\zeta}(n^{1/\zeta}+1)}\right)\mathrm{d}n$$
$$= \frac{1}{2\zeta^2}\mathrm{e}^{1/x^{\zeta}}[\mathrm{E}_1(1/x^{\zeta})]^2 + \int_1^{\infty}\frac{\ln(n)}{n+1}\left(1 - \frac{n^{(\zeta-1)}(n+1)}{1+n^{\zeta}}\right)\mathrm{d}n \qquad \text{(A.9)}$$

$$\int_1^{\infty}\frac{\ln(n)}{n+1}\left(1 - \frac{n^{(\zeta-1)}(n+1)}{1+n^{\zeta}}\right)\mathrm{d}n$$
$$= -\int_0^1\frac{\ln(n)}{n(n+1)}\left(1 - \frac{1+n}{1+n^{\zeta}}\right)\mathrm{d}n$$
$$= -\int_0^1\frac{\ln(n)}{n(n+1)}\left(1 - (1+n)\sum_{m=0}^{\infty}(-1)^m n^{\zeta m}\right)\mathrm{d}n$$

$$=\left(\int_0^1 \sum_{m=1}^{\infty} (-1)^m n^{(m\zeta-1)} \ln(n)\, \mathrm{d}n + \int_0^1 \frac{\ln(n)}{n+1}\, \mathrm{d}n\right)$$

$$=-\left(\sum_{m=1}^{\infty} (-1)^m/(m\zeta)^2 + (\pi^2/12)\right) = -\frac{\pi^2}{12}\left(1-\frac{1}{\zeta^2}\right). \qquad \text{(A.10)}$$

Provided $\zeta \geqslant 1$. Since $x \gg 1$ we may replace the exponential integral by its small-order term to yield:

$$S_2 \simeq B^1 + \tfrac{1}{4}(\ln x - \gamma/\zeta)^2 + \pi^2/24\zeta^2 \qquad \text{(A.11)}$$

where

$$B^1 = \lim_{N\to\infty}\left(\sum_{n=1}^{N} [\psi(n)/(2n+1)] - \tfrac{1}{4}(\ln N)^2\right) = -0{\cdot}589\,1150 \qquad \text{(A.12)}$$

was calculated directly.

From (A.5) and (A.11) we note that

$$S_2 \simeq [S_1 - (\tfrac{1}{2}\gamma + \ln 2 - 1)]^2 + (\pi^2/24\zeta^2) + B_1 \qquad \text{(A.13)}$$

which has been found by numerical calculation to be more accurate than (A.11) for large, but finite, x.

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